RIEMANN FUNCTIONS FOR A LINEAR SYSTEM OF HYPERBOLIC FORM

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ABSTRACT

Functions are defined which permit the solution of a special hyperbolic system to be expressed as a quadratur of its initial data over the initial surface.

1. **Introduction.** In this paper Riemann functions (R.F.) are defined for systems of partial differential equations of the type

$$L(U) \equiv \frac{\partial}{\partial x} U - AU = 0 \tag{1.1}$$

where

$$U \equiv (U^1, \cdots, U^N); \; A = (a_{ij}), \; 1 \leqq i,j \leqq N$$

and

$$\frac{\partial}{\partial x} U \equiv \left(\frac{\partial U^1}{\partial x_1}, \cdots, \frac{\partial U^N}{\partial x_N} \right)$$

We will use an elementary method to define a set of functions (the R.F.) that enable the value of U^i at an arbitrary point P, not on the initial surface θ, to be expressed as a quadrature of its Cauchy data on θ.

The subject dealt with here is similar in its nature to that in [1, 2]; further references can be found in [2].

2. **Auxiliary concepts and notation.** To achieve simplicity we chose without loss of generality θ to be the hyperplane

$$\theta\colon \sum_{i=1}^{N} x_i = 1 \tag{2.1}$$

and the point P to be the origin. Some additional concepts and notation needed for our definition of the R.F. are introduced in this section.

Many of our constructions are associated with subsets $\{i_1, i_2, \cdots\}$ of the first N positive integers. Since these constructions are in no way dependent on the

Received August 26, 1966.

particular properties of the integers composing the subset it is notationally convenient to chose some fixed but arbitrary subset and symbolize it by a letter. Hence $H \equiv \{i_1, \cdots, i_k\}$ is a fixed subset consisting of k integers i_j such that $1 \leqq i_j \leqq N$, $j = 1, \cdots, k$; and $\bar{H}$ is the set theoretic compliment of H in the first N positive numbers. Also needed are two types of sets that are derived from H. The $N-k$ sets H_p are defined by

$$H_p \equiv \{H \oplus p\}, \quad p \in \bar{H};$$

and for each $p \in \bar{H}$ the $k+1$ sets H_{pq} are defined by

$$H_{pq} \equiv \{H_p \ominus q), \quad q \in H_p$$

where $\oplus$ and $\ominus$ are set theoretic union and subtraction. From these definitions it is seen that $H_{pp} = H$.

We associate certain volumes in R^N with the subsets H, H_p and H_{pq}. Thus letting H^* denote any of these subsets or any subset of the first N positive integers we define the volume $V[H^*]$ by

$$V[H^*] \equiv \left\{ x = (x_1, \cdots, x_N) \,\middle|\, x_j \geqq 0, \sum_{j=1}^{N} x_j = \sum_{j \in H^*} x_j \leqq 1 \right\}$$

and the subset of θ, $I[H^*]$, by

$$I[H^*] \equiv \left\{ x = (x_1, \cdots, x_N) \,\middle|\, x_j \geqq 0, \sum_{j=1}^{N} x_j = \sum_{j \in H^*} x_j = 1 \right\}.$$

These definitions are motivated by geometric considerations. Note that $I[H_p]$ is the projection of $V[H_{pq}]$ along the axis x_q onto θ. Furthermore, $V[H_p]$ is the volume contained "between" $V[H_{pq}]$ and $I[H_p]$ that is "swept out" when $V[H_{pq}]$ is projected along x_q onto θ. These sets and ideas can be readily visualized in 3-dimensional space. It should be noted that $V^t \equiv V[i_1, \cdots, i_N]$ is independent of the permutation of $i_1, \cdots, i_N$ and consists of all points enclosed by θ and the hyperplanes $x_i = 0$, $i = 1, \cdots, N$.

The definition of the R.F. employ auxiliary systems of equations that are derived from (1.1). Thus the system

$$L[H](U) = 0 \tag{2.2}$$

is derived from (1.1) by striking from it every lth row and column where $l \in \bar{H}$. We also use subsidiary systems formed from the system $L[H](U) = 0$. These subsidiary systems, called the progenitors of $L[H](U) = 0$, are the $(N-k)$ systems,

$$L[H_p](U) = 0, \tag{2.3}$$

defined for $p \in \bar{H}$. Furthermore $L[H](U)_q \equiv I_{x_n} U^q - \sum_{l \in H} a_{ql} U^l$, $q \in H$.

3. **The definition of the Riemann functions.** The fundamental relation that we introduce below is basic for a certain reduction procedure that is to be used in the definition of the R.F. This relation expresses a particular quadrature of an arbitrary system in terms of quadratures of its progenitor systems and additional terms that can be evaluated from the given Cauchy data on the hyperplane θ. We derive this fundamental relation in two steps.

When Green's identities for the volumes $V[H_p]$, $p \in \bar{H}$ are added together we get

$$\sum_{p \in \bar{H}} \int_{V[H_p]} \sum_{q \in H_-} V^{pq} L[H_p](U)_q dx \cdots + \sum_{p \in \bar{H}} \int_{V[H_-]} \sum_{q \in H_-} U^q L^*[H_p](V^p)_q dx \cdots$$

$$(3.1) \qquad = \sum_{p \in \bar{H}} \sum_{q \in H} \left[\int_{V[H_{pq}]} U^q V^{pq} dx \cdots - \int_{I[H_p]} U^q V^{pq} dx \cdots \right]$$

where $L^*[H_p]$ is the adjoint operator to $L[H_p]$ and the vector valued functions $V^p \equiv (V^{pq})$ $p \in \bar{H}, q \in H_p$ are defined in $V[H_p]$.

The second step in deducing the fundamental relationship is to specify the vector valued functions $V^p = (V^{pq})$ as

$$\text{i)} \quad V^p \text{ is a solution of } L^*[H_p](V^p) = 0 \text{ in } V[H_p]$$

$$(3.2) \qquad \text{ii)} \quad V^{pq} = 0 \text{ on } V[H_{pq}] \text{ for } q \in H_p \text{ but } q \neq p$$

$$\text{iii)} \quad V^{pp} = \sum_{l \in H} W^l a_{lp} \text{ on } V[H_{pp}] = V[H]$$

where W^l, $l \in H$ are unspecified functions defined in $V[H]$.
Since from (1.1)

$$(3.3) \qquad \sum_{p \in \bar{H}} \int_{V[H]} \sum_{l \in H} W^l a_{lp} U^p dx \cdots = \int_{V[H]} \sum_{l \in H} W^l L[H](U)_l dx \cdots$$

we get by using (3.2) in (3.1) that

$$(3.4) \qquad \int_{V[H]} \sum_{l \in H} W^l L[H](U)_l dx \cdots = \sum_{p \in \bar{H}} \int_{V[H_-]} \sum_{q \in H_-} V^{pq} L[H_p](U)_q dx \cdots$$

$$- \sum_{p \in \bar{H}} \sum_{q \in H_-} \int_{I[H_-]} U^q V^{pq} dx \cdots$$

which is the fundamental relation.

It expresses the integral

$$(3.5) \qquad \int_{V[H]} \sum_{l \in H} W^l L[H](U)_l dx \cdots$$

of the system $L[H](U) = 0$ in terms of exactly similar type integrals of its progenitor systems $L[H_p](U) = 0$, $p \in \bar{H}$ and certain quadratures over $I[H_p] \subset \theta$ of functions which are determined on θ. Since the integrals on the right hand side of (3.4) are of the exact same form as that in (3.5) these integrals themselves can be re-expressed in terms of their own progenitors and further quadratures over θ. The process can be repeated until every integral like (3.5) is expressed in terms of the original progenitor, (1.1), and quadratures over subsets of θ.

We now show how a typical component U^1 of U can be expressed at the point $(0,\cdots,0)$ as a linear functional of its Cauchy data and certain additional functions (the R.F.) over θ. By Green's identity

$$(3.6)\quad VU^1(0,\cdots,0) = VU^1(1,0,\cdots,0) + \int_{(1,0,\ldots,0)}^{(0,\ldots,0)} [VL[1](U)_1 + U^1L^*[1](V)_1]dx,$$

where $L^*[1]$ is the adjoint to $L[1]$. After chosing $V(x_1,0,\cdots,0)$ to satisfy

$$(3.7)\qquad L^*[1](V) = 0$$

and

$$V(0,\cdots,0) = 1$$

equation (3.6) becomes

$$(3.8)\qquad U^1(0,\cdots,0) = VU^1(1,0,\cdots,0) + \int_{(1,0,\ldots,0)}^{(0,\ldots,0)} VL[1](U)_1\, dx_1$$

The integral term in (3.8) is of the form (3.5) and hence by using the fundamental relation (3.4) it can be expressed in terms of integrals of known quantities evaluated over subsets of θ and an integral of the ultimate progenitor system (1.1) integrated over $V^t \equiv V[1,\cdots,N]$. Thus

$$U^1(0,\cdots,0) = \{\text{expressions involving the } V\text{'s defined in (3.2) and (3.7)}\} + \int_{V^t} \sum_{j=1}^{N} V^jL(U)_j\, dx_1 \cdots dx_N$$

Because $L(U) = 0$ the last term in (3.9) vanishes and hence $U^1(0,\cdots,0)$ has been expressed as a linear functional over θ with the aid of the functions defined in (3.2) and (3.7); hence these functions are the R.F, for the system (1.1).

References

1. L. Bianchi *Sulla estensione del metodo di Riemann alle equazioni lineari a derivate parzioli d'ordine superiore Noto* III: Rend. Accad. Lincei, Ser. 5, 4 (1895), p. 133–142.

2. H. M. Sternberg and J. B. Diaz, *Computation of the Riemann function for the operator* $\partial^n/\partial x_1\cdots\partial x_n + a(x_1\cdots x_n)$ Mathematics of Computation, **19** (1965), pp. 562–569.

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